Invertible cells in weak ω-categories

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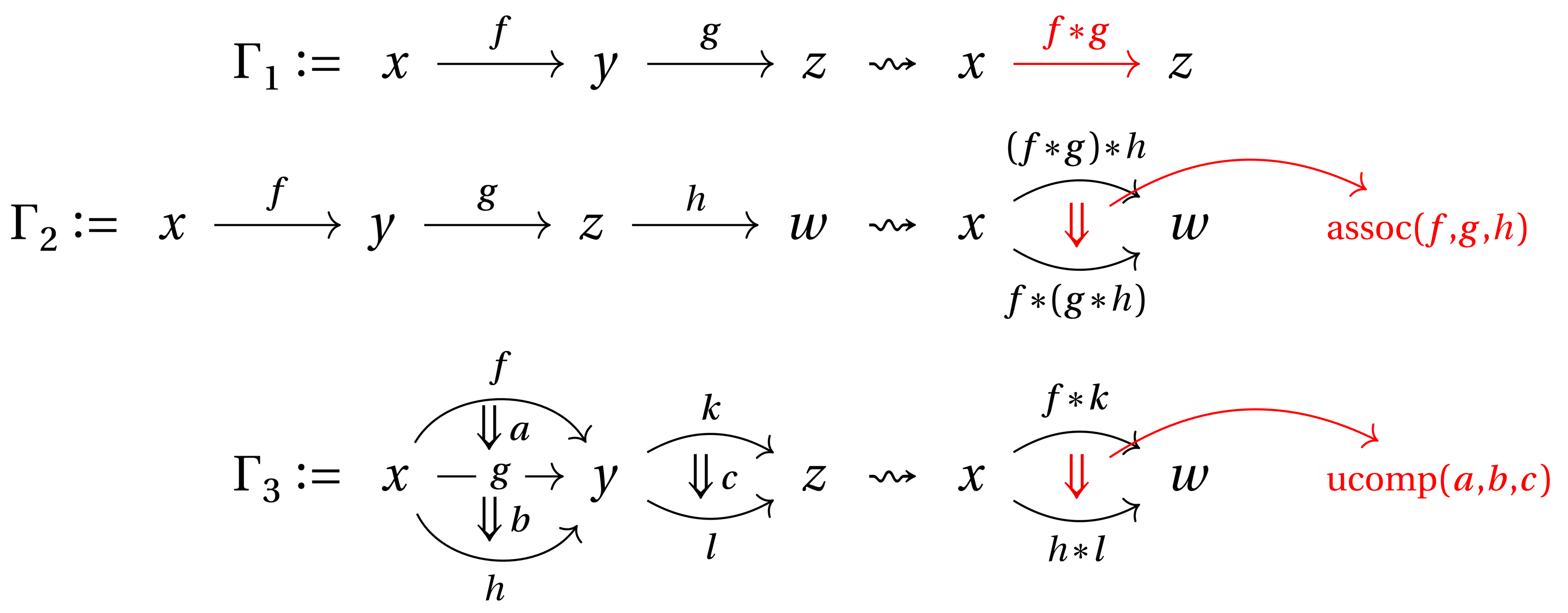
Weak ω-categories

Globular, weak ω-categories [2, 8, 9] are an algebraic model of (∞,∞)-categories. They are collections of cells equipped with an essentially unique way to compose pasting diagrams.



The operations of ω-categories are generated by

- Each choice of compositions for the source and target of a pasting diagram yields a composition operation for the pasting diagram.
- Any two operations over the same pasting diagram are equivalent.

The type theory Catt

Catt [6] is a dependent type theory, whose models are ω-categories. Catt has a type of objects $\star$ and a type of (higher) morphisms $u \to v$ for any pair of terms u, v of the same type. Terms are either variables, or constructed by one of the following rules:

$$\frac{\Gamma \vdash_{ps} \quad \partial^{-}\Gamma \vdash u : A \quad \partial^{+}\Gamma \vdash v : A}{\Gamma \vdash \mathrm{comp}(\Gamma, u \to v) : u \to v} \qquad \frac{\Gamma \vdash_{ps} \quad \Gamma \vdash u : A \quad \Gamma \vdash v : A}{\Gamma \vdash \mathrm{coh}(\Gamma, u \to v) : u \to v}$$

: pasting diagram   : source/target operations

The source and target operations are required to to use all available variables. Contexts of this type theory correspond to finite computads [1, 5, 3], that is ω-categories freely generated by a finite number of generators.

$$f * g := \mathrm{comp}(\Gamma_1, x \to z)$$
$$\mathrm{assoc}(f,g,h) := \mathrm{coh}(\Gamma_2, (f*g)*h \to f*(g*h))$$
$$\mathrm{ucomp}(a,b,c) := \mathrm{comp}(\Gamma_3, f*k \to h*l)$$

We can also build the Eckmann-Hilton operation

$$(x : \star)(a, b : \mathrm{id}\, x \to \mathrm{id}\, x) \vdash \mathrm{eh} : \; a * b \to b * a.$$

Coinductive invertibility

A cell $f : a \to b$ is defined to be invertible [4] if there exists an inverse cell $f^- : b \to a$ together with invertible cancellation witnesses

$$u_f : f * f^- \to \mathrm{id}(a) \qquad v_f : f^- * f \to \mathrm{id}(b)$$

⚠ This is a coinductive definition. To show that a cell is invertible, one needs to provide an infinite amount of higher cells.

Contributions

Our main contributions are the following:

- Every coherence cell is invertible.
- A composite of cells is invertible when the cells of maximal dimension are invertible.
- In a finite-dimensional computad, the converse also holds. More precisely, a cell is invertible if and only if it only uses generators of lower dimension.

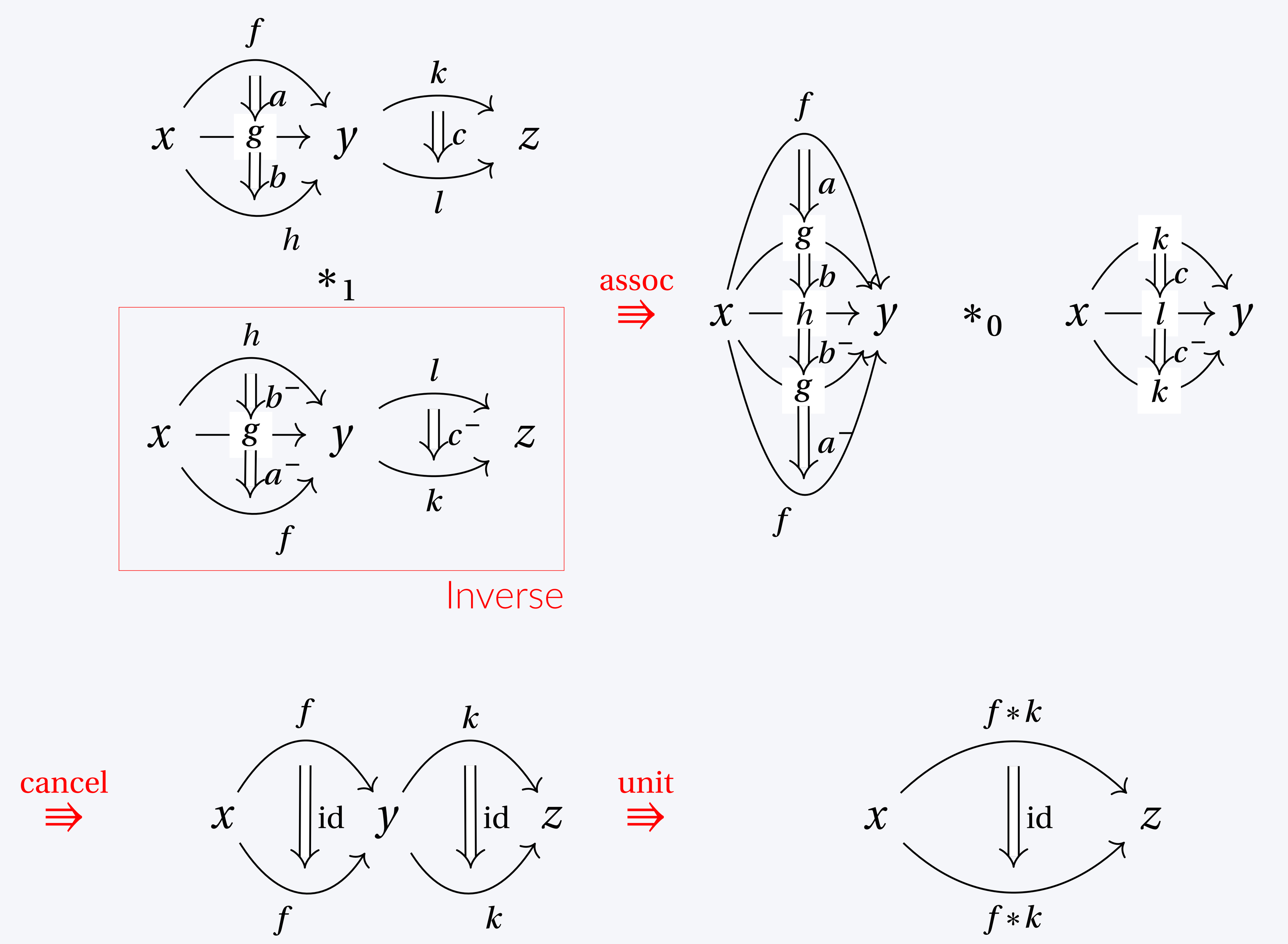
These results extend those of Fujii, Hoshino and Maehara [7].

We provide an algorithm to compute an inverse and cancellation witnesses for coherence cells and composites of invertible cells.

- Inverses and witnesses of coherence cells can be obtained from the coherence rule.
- Inverses of composite cells can be obtained by composing inverses in the opposite order. Cancellation witnesses can be built using associators, unitors and composites of cancellation witnesses.

Example: Inverse of an unbiased composite

Provided a, b, c are invertible, an inverse and cancellation witness of $\mathrm{ucomp}(a,b,c)$ can be computed by:



Application

These techniques can be used to significantly reduce the proof obligations in the proof assistant Catt for ω-categories, maintained by the first author. The Eckmann-Hilton cell is invertible. Using automatic computation of witnesses, we reduced the size of the program defining the cancellation witness of the Eckamnn-Hilton by 95% of its size.

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