Computads for generalised signatures

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UNIVERSITY OF CAMBRIDGE

ONASSIS FOUNDATION

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Free Algebras

Question

What does it mean for a mathematical structure to be free?

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Free Algebras

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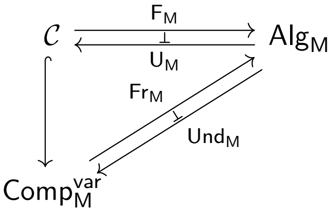

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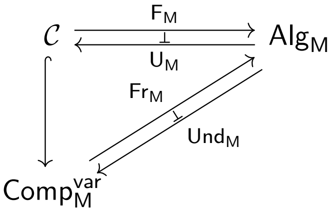

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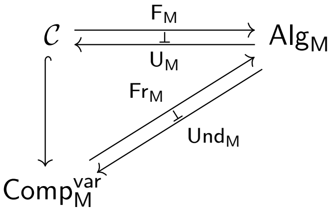

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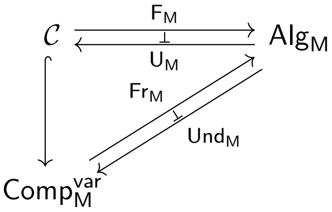

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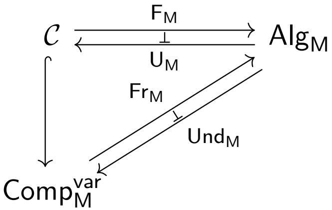

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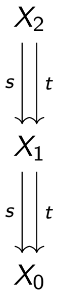
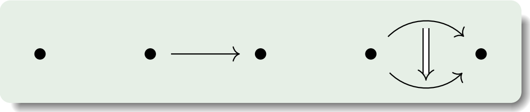
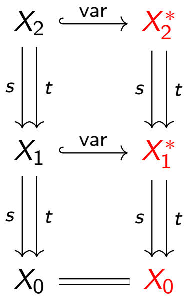


- A 2-graph X is a diagram satisfying the *globularity conditions*

$$ss = st \qquad ts = tt$$



Computads for 2-categories



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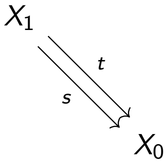

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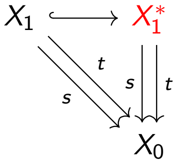

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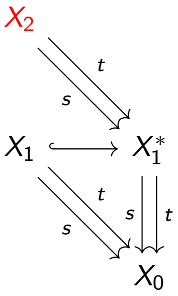

- A 2-computad X is a diagram satisfying the *globularity conditions*

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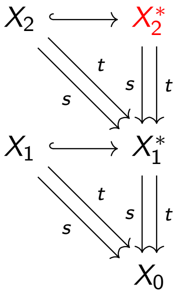

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Monads are mathematical structures. When is a monad free?

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It should correspond to a theory with no equations when $\mathcal{C} = [\mathcal{I}^{\mathsf{op}}, \mathsf{Set}]$.

- weak ω-categories
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It should correspond to a theory with no equations when $\mathcal{C} = [\mathcal{I}^{\mathrm{op}}, \mathrm{Set}]$.

- weak ω-categories
- algebraic Kan complexes / quasicategories

Definition

A signature[4] over $\mathcal{C} = [\mathcal{I}^{\mathrm{op}}, \mathrm{Set}]$ is a presheaf $\Sigma \in \mathcal{C}$ of function symbols with arity functions $B_\bullet : \Sigma_i \to \mathrm{ob}\,\mathcal{C}$ compatible with morphisms in $\mathcal{I}$.

[4] Bourke and Garner, "Monads and theories"

Algebras of a signature

Definition

A Σ-algebra $\mathbb{X}$ is a presheaf X equipped with functions

$$f^{\mathbb{X}} : \mathcal{C}(B_f, X) \to X_i$$

for $f \in \Sigma_i$ compatible, satisfying that $(\delta^* f)^{\mathbb{X}} = \delta^* f^{\mathbb{X}}$ for $\delta : j \to i$.

- The forgetful functor $\mathrm{U}_\Sigma : \mathrm{Alg}_\Sigma \to \mathcal{C}$ is strict monadic.
- The free monad on Σ is $\mathrm{M}_\Sigma = \mathrm{U}_\Sigma \mathrm{F}_\Sigma$.

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Problem

Weak ω-categories are not algebras of a signature, since the source of the associator is not an 1-dimensional function symbol, but a composite of them.

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Direct Categories

Definition

A direct category $\mathcal{I}$ is a small category equipped with a dimension function $\dim : \mathrm{ob}\,\mathcal{I} \to \mathrm{Ord}$ to the class of ordinals such that $\dim j < \dim i$ for every non-identity morphism $\delta : j \to i$.

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$$[0] \underset{t}{\overset{s}{\rightrightarrows}} [1] \underset{t}{\overset{s}{\rightrightarrows}} [2] \underset{t}{\overset{s}{\rightrightarrows}} \cdots \qquad \begin{aligned} ss &= ts \\ st &= tt \end{aligned}$$

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- The category Δ_{inj} of simplices and face maps.

$$[0] \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} [1] \underset{\delta_2}{\overset{\delta_0}{\rightrightarrows}} [2] \underset{\delta_3}{\overset{\delta_0}{\rightrightarrows}} \cdots \qquad \delta_i \delta_j = \delta_{j+1} \delta_i \quad (i \leq j)$$

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Notation

We denote by $\mathcal{I}_\alpha$ the full subcategory on objects of dimension at most α with the obvious dimension function.

The inductive data

We define by transfinite recursion on $\alpha \leq \sup\{\dim i \;:\; i \in \mathcal{I}\}$

- a class $\mathsf{Sig}_\alpha(\mathcal{I})$ of (generalised) signatures of dimension α,

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$$[\mathcal{I}_\alpha^{\mathsf{op}}, \mathsf{Set}] \underset{\mathsf{Term}_\Sigma}{\overset{\mathsf{Cptd}_\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Comp}_\Sigma$$

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  - an arity $B_f \in [\mathcal{I}^{\mathrm{op}}, \mathrm{Set}]$
  - for non-identity $\delta : j \to i$, a boundary term

$$\delta^* f \in \mathrm{Term}_{\Sigma_{\dim j}, j}\, \mathrm{Cptd}_{\Sigma_{\dim j}}(\mathrm{tr}_{\dim j}\, B_f)$$

compatible with morphisms in $\mathcal{I}$.

The restriction functions are the obvious projections.

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Each presheaf $X : \mathcal{I}_\alpha^{\mathrm{op}} \to \mathrm{Set}$ defines a computad where the sets of generators are given by

$$V_i^{\mathrm{Cptd}_\Sigma X} = X_i$$

and the gluing functions by the functions $\delta^* : X_i \to X_j$.

Terms

Definition

The lower dimensional terms of a computad C are those of C_β. The set of terms $\mathrm{Term}_{\Sigma,i}\, C$ for $\dim i = \alpha$ is generated inductively by two rules:

- There exists a term $\mathrm{var}\, v$ for every generator $v \in V_i^C$.
- There exists a term $\mathrm{comp}[f, \tau]$ for $f \in \Sigma_i$ and $\tau : \mathrm{Cptd}_\Sigma\, B_f \to C$

The boundaries of top dimensional terms are given by the gluing functions and the chosen boundary terms of the function symbols:

$$\delta^*(\mathrm{var}\, v) = \phi_\delta^C(v)$$
$$\delta^*(\mathrm{comp}[f, \tau]) = \mathrm{Term}_{\Sigma_{\dim j}, j}(\tau_{\dim j})(\delta^* f)$$

Morphisms of computads

Definition

A morphism $\sigma : D \to C$ consists of

- morphisms $\sigma_\beta : D_\beta \to C_\beta$ for $\beta < \alpha$ such that $\mathrm{tr}_\gamma^{\Sigma_\beta} \sigma_\beta = \sigma_\gamma$,
- functions $\sigma_i : V_i^D \to \mathrm{Term}_{\Sigma,i}(C)$ for $\dim i = \alpha$ satisfying for $\delta : j \to i$ that

$$\begin{array}{ccc} V_i^D & \xrightarrow{\sigma_i} & \mathrm{Term}_{\Sigma,i}\, C \\ \downarrow{\scriptstyle \phi_\delta^D} & & \downarrow{\scriptstyle \delta^*} \\ \mathrm{Term}_{\Sigma_{\dim j},j}\, D & \xrightarrow{\sigma_{\dim j}} & \mathrm{Term}_{\Sigma_{\dim j},j}\, C \end{array}$$

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We will say that $\sigma : D \to C$ is variable-to-variable when σ_β are variable-to-variable and σ_i factors via var.

Composition

Composition and the action of a morphism $\sigma : C \to D$ on terms are defined mutually recursiely. The action on terms is given by

$$\begin{aligned} \mathsf{Term}_{\Sigma,i}(\sigma)(\mathsf{var}\ v) &= \sigma_i(v) \\ \mathsf{Term}_{\Sigma,i}(\sigma)(\mathsf{comp}[f,\tau]) &= \mathsf{comp}[f,\sigma\circ\tau]. \end{aligned}$$

Composition of morphisms is given by

$$\sigma\circ\tau = (\sigma_\beta\circ\tau_\beta, \mathsf{Term}_{\Sigma,i}(\sigma)\circ\tau_i)$$

Functoriality of Term_Σ and associativity of composition are proven mutually inductively. Identities are given by the inclusions var of variables to terms.

The term adjunction

- The unit includes generators into terms

$$\eta_{\Sigma,X} : X \to \mathsf{Term}_\Sigma \, \mathsf{Cptd}_\Sigma \, X$$
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$$\varepsilon_{\Sigma,C} : \mathsf{Cptd}_\Sigma \, \mathsf{Term}_\Sigma \, C \to C$$
$$\varepsilon_{\Sigma,C,\beta} = \epsilon_{\Sigma_\beta, C_\beta}$$
$$\varepsilon_{\Sigma,C,i} : V_i^{\mathsf{Cptd}_\Sigma \, \mathsf{Term}_\Sigma \, C} = \mathsf{Term}_\Sigma \, C$$

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The term monad $(\mathsf{M}_\Sigma, \eta_\Sigma, \mu_\Sigma)$ is the monad on $[\mathcal{I}_\alpha^{\mathsf{op}}, \mathsf{Set}]$ induced by the term adjunction. The category Alg_Σ is the category of M_Σ-algebras.

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Algebras $\mathbb{X} = (X, u^{\mathbb{X}})$ are pairs of a presheaf and a morphism

$$u^{\mathbb{X}} : \mathsf{Term}_\Sigma \, \mathsf{Cptd}_\Sigma \, X \to X$$

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satisfying two axioms. They gives rise for every $f \in \Sigma_i$ to functions

$$f^{\mathbb{X}} : [\mathcal{I}_\alpha^{\mathsf{op}}, \mathsf{Set}](B_f, X) \to X_i$$
$$f^{\mathbb{X}}(\tau) = u^{\mathbb{X}}(\mathsf{comp}[f, \mathsf{Cptd}_\Sigma \, \tau])$$

and they are determined by them.

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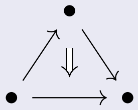
Kan complexes

Notation

We will let $\mathsf{ssSet} = [\Delta_{\mathsf{inj}}^{\mathsf{op}}, \mathsf{Set}]$ and

- $\Delta[n]$ the representable semisimplicial set



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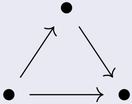

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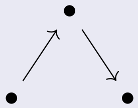

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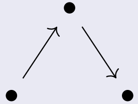

Definition

An algebraic semisimplicial Kan complex $\mathbb{X}$ is a semisimplicial set X together with for $n > 0$ and $0 \leq k \leq n$ a choice of lift for

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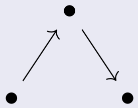

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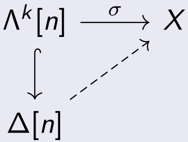

$$\text{face}_{k,n} : \text{ssSet}(\Lambda^k[n], X) \to X_{n-1}$$

$$\text{fill}_{k,n} : \text{ssSet}(\Lambda^k[n], X) \to X_n$$

Kan Complexes

Kan complexes are algebras of a signature defined by

$$\Sigma_{\mathsf{Kan},[n]} = \{\mathsf{face}_{n+1,k} \ : \ 0 \le k \le n+1\} \cup \{\mathsf{fill}_{k,n} \ : \ n > 0,\ 0 \le k \le n\}$$

$$B_{\mathsf{face}_{k,n+1}} = \Lambda^k[n+1] \qquad \delta^* \,\mathsf{face}_{k,n+1} = \mathsf{var}(\delta_k \delta)$$

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Kan Complexes

Kan complexes are algebras of a signature defined by

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Remark

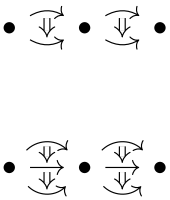
Algebraic semisimplicial Kan complexes carry a model structure equivalent to spaces[5]. We will discuss its cofibrations shortly.

[5] Bourke and Henry, *Algebraically cofibrant and fibrant object revisited*

Globular Higher Categories

Definition

We let Glob $= [\mathbb{G}^{\mathrm{op}}, \mathrm{Set}]$ the category of globular sets. We let Bat the family of globular sets representing the strict ω-category monad.



Globular Higher Categories

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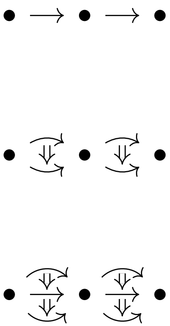
The signature $\Sigma_{\omega\,\mathrm{cat}}$ for weak ω-categories[6]

- has no operations of dimension 0,
- has as operations of dimension $n+1$ triples (B, a, b) where
  - $B \in \mathrm{Bat}$ of dimension at most $n+1$,
  - $a, b \in \mathrm{Term}_n(B)$ that "cover" the n-dimensional source and target of B respectively, and have common source and target.



[6] Dean et al., *Computads for weak ω-categories as an inductive type*

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Computads as presheaves

Lemma

The subcategory $\mathsf{Comp}_\Sigma^{\mathsf{var}}$ *of variable-to-variable morphisms has a terminal computad* $\mathbb{1}_\Sigma$*.*

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$$V_i^C \cong \coprod_{p \in V_i^{\mathbb{1}_\Sigma}} \mathsf{Comp}_\Sigma^{\mathsf{var}}(|p|, C).$$

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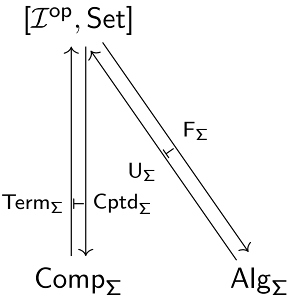
Theorem

Let Plex_{Σ} *the full subcategory of* $\mathsf{Comp}^{\mathsf{var}}_{\Sigma}$ *on the computads* $|p|$. *The subcategory inclusion induces an equivalence*

$$\mathsf{Comp}^{\mathsf{var}}_{\Sigma} \cong [\mathsf{Plex}^{\mathsf{op}}_{\Sigma}, \mathsf{Set}]$$

Free Algebras



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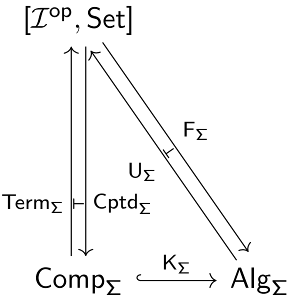
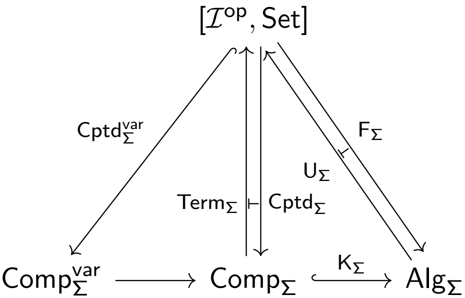


- K_Σ is fully faithfull

Free Algebras



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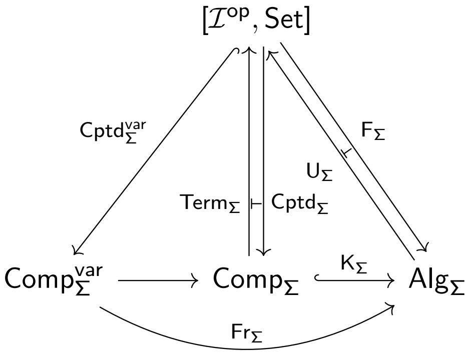

- K_{Σ} is fully faithfull
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$$V_i^C \to X_i$$

compatible with the gluing functions.

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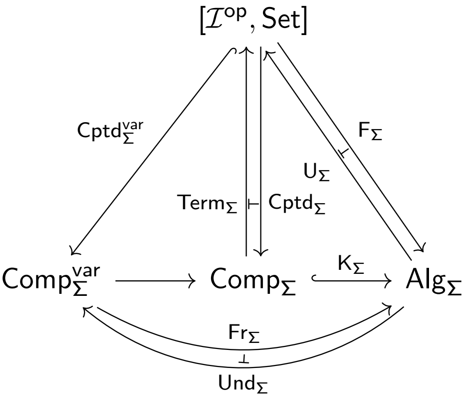

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Computads as cofibrant algebras

Let $\mathbb{D}^i = \mathrm{Fr}_\Sigma \mathcal{I}(-, i)$ and $\partial\mathbb{D}^i$ its boundary. The inclusions $\partial\mathbb{D}^i_\Sigma \subseteq \mathbb{D}^i_\Sigma$ cofibrantly generate a weak factorisation system on the category of algebras.

Remark

This is the (cofibration, trivial fibration) wfs for Kan complexes and for strict ω-categories.

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The comonad $\mathsf{Fr}_\Sigma \mathsf{Und}_\Sigma : \mathsf{Alg}_\Sigma \to \mathsf{Alg}_\Sigma$ *is the universal cofibrant replacement*[7] *of the wfs.*

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[7] Garner, "Homomorphisms of higher categories"

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