

Naturality for weak ω -categories

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Grothendieck's homotopy hypothesis

- A classical result in topology is the equivalence between n -truncated spaces up to homotopy and strict n -groupoids up to equivalence for $n \leq 2$ induced by the fundamental n -groupoid functor

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Weak ω -groupoids are equivalent to spaces up to homotopy.

Definition

A *globular set* is a diagram of sets of the form

$$X_0 \begin{array}{c} \xleftarrow{\text{src}} \\ \xleftarrow{\text{tgt}} \end{array} X_1 \begin{array}{c} \xleftarrow{\text{src}} \\ \xleftarrow{\text{tgt}} \end{array} X_2 \begin{array}{c} \xleftarrow{\text{src}} \\ \xleftarrow{\text{tgt}} \end{array} \dots$$

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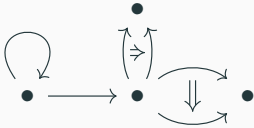
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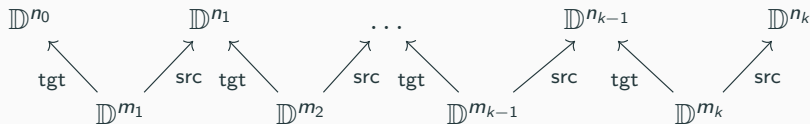
$$\text{tgt} \circ \text{src} = \text{tgt} \circ \text{tgt}$$

The disk $\mathbb{D}^n$ is determined by $\text{Glob}(\mathbb{D}^n, X) \cong X_n$



Pasting diagrams

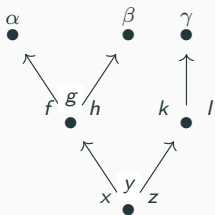
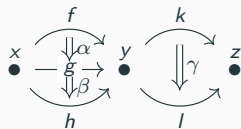
- Pasting diagrams* are a family of contractible globular sets, that can be written as iterated pushouts of disks of the form:



- They admit a well-defined notion of dimension and of boundary.
- These are the arities for the operations of ω -categories in the sense of Batanin and Leinster, or equivalently of Grothendieck and Maltsiniotis.

Representations of pasting diagrams

$$\mathbb{D}^2 \coprod_{\mathbb{D}^1} \mathbb{D}^2 \coprod_{\mathbb{D}^0} \mathbb{D}^2$$



Towards weak ω -categories

Guiding principle

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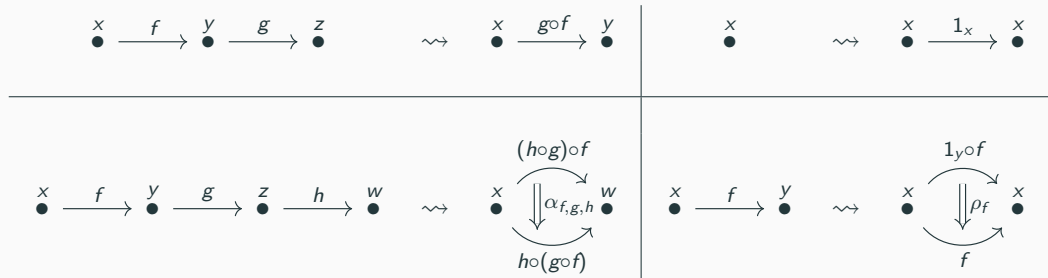
1. Given a way to compose the source and target of a pasting diagram, there is a way to compose the entire diagram.
2. Given two ways to compose a pasting diagram, there exists a way to relate them.

Towards weak ω -categories

Guiding principle

Every pasting diagram in a weak ω -category should be composed “uniquely”:

1. Given a way to compose the source and target of a pasting diagram, there is a way to compose the entire diagram.
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Definition

A weak ω -category consists of a globular set X equipped with a family of operations

$$\mathrm{coh}_{B,u \rightarrow v}: \mathrm{Glob}(B, X) \rightarrow X_{n+1}$$

where B is a pasting diagram of dimension at most $n + 1$ and u, v are a pair of n -cells of the free weak ω -category on B subject to a *fullness* condition. A strict functor $F: X \rightarrow Y$ is a morphism of globular sets that commutes with the operations.

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- This is a definition that is recursive on the dimension of the operations.
- The fullness condition corresponds to the guiding principles mentioned before. Removing it gives a groupoidal theory.

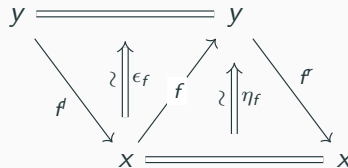
Invertibility in ω -categories

Definition

A morphism $f: x \rightarrow y$ in an ω -category is coinductively invertible if there exist $f^\sharp, f^\flat: y \rightarrow x$ and coinductively invertible

$$\epsilon_f: f \circ f^\sharp \Rightarrow 1_x$$

$$\eta_f: 1_y \Rightarrow f^\flat \circ f$$

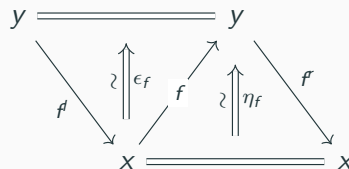


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Theorem

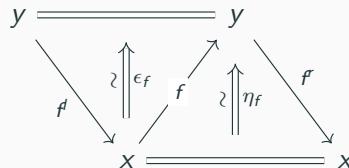
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Theorem

A cell $\text{coh}_{B,u \rightarrow v}(f) \in X_n$ is invertible when $f(p)$ is invertible for all $p \in B_n$.

- Identities are invertible
- All coherences are invertible ($\dim B < n$)
- A composite is invertible when it's top-dimensional arguments are ($\dim B = n$)

Definition

A strict functor $f: X \rightarrow Y$ is a trivial fibration (resp. equivalence) when:

- it is (resp. essentially) surjective on objects, and
- $f_{x,y}: X(x, y) \rightarrow Y(fx, fy)$ is a trivial fibration (resp. equivalence) for all $x, y \in X_0$

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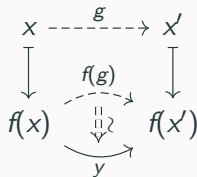
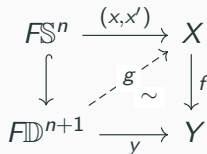
Equivalences

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- $\{\text{Trivial fibrations}\} = \{FS^{n-1} \subseteq F\mathbb{D}^n\}^\natural$ for $S^{n-1} = \mathbb{D}^n \setminus \{d_n\}$
- Equivalences satisfy a weaker lifting property:



Model structure on ω -categories

Conjecture

There exists a model structure on the category of weak ω -categories with these equivalences and trivial fibrations.

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- We are missing the cylinder objects for cofibrant weak ω -categories

$$\begin{array}{ccccc} X \amalg X & \xrightarrow{(\iota^-, \iota^+)} & \text{Cyl } X & \xrightarrow{\sim} & X \\ & \searrow & \text{---} & \nearrow & \\ & & (\text{id}, \text{id}) & & \end{array}$$

$$\text{Cyl } \mathbb{D}^1 = \begin{array}{ccccc} x^- & \xrightarrow{f^-} & & y^- \\ \downarrow \wr & \swarrow \vec{f} & \searrow \wr & \downarrow \wr & \\ x^+ & \xrightarrow{f^+} & & y^+ \end{array}$$

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- A 0-computad C is a set and $FC = C \cdot \mathbb{D}^0$
- An $(n+1)$ -computad $C = (C_n, V, \text{bdry})$ consists of

- an n -computad C_n
- a set of generators V_{n+1}
- a morphism $\text{bdry} : V_{n+1} \cdot FS^n \rightarrow FC_n$.

$$\begin{array}{ccc}
 V_{n+1} \cdot FS^n & \xrightarrow{\text{bdry}} & FC_n \\
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- A computad $C = (C_n)_{n \in \mathbb{N}}$ is a compatible sequence of n -computads and

$$FC = \text{colim}_{n \in \mathbb{N}} FC_n$$

The inductive framework

We define recursively on the dimension $n \in \mathbb{N}$ of a computad C , for every upclosed set of generators $X \subseteq \coprod_n V_n^C$, the following:

- a computad $C \uparrow X$ and two strict functors:

$$\iota_{C,X}^-, \iota_{C,X}^+ : C \rightarrow C \uparrow X$$

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- For a cell $c \in (FC)_m$ a new cell

$$c \uparrow X \in F(C \uparrow X)_{m+1}$$

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Remark

$$C \uparrow (\coprod V_n^C) = \text{Cyl } C$$

Naturality of computads

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- $C \uparrow X$ for $C = (C_n, V, \text{bdry})$ is built by first forming the pushout:

$$\begin{array}{ccc}
 V \cdot FS^n + (X \cap V) \cdot FS^n & \hookrightarrow & V \cdot F\mathbb{D}^{n+1} + (X \cap V) \cdot F\mathbb{D}^{n+1} \\
 \downarrow \iota^- \circ \text{bdry} + \iota^+ \circ \text{bdry} & \lrcorner & \downarrow \\
 C_n \uparrow X_{\leq n} & \dashrightarrow & (C \uparrow X)_{n+1}
 \end{array}$$

and then adding $X \cap V$ as set of $(n+2)$ -generators with

$$\text{bdry}(\vec{x}) = \iota^-(x) \curvearrowright \iota^+(x) = \text{src}^2(x) \uparrow X$$

Naturality of functors and cells

The functor $f \uparrow X: (D \uparrow f^{-1}X) \rightarrow (C \uparrow X)$ is determined by:

$$\begin{aligned}(f \uparrow X)(v) &= f(v) \text{ for } v \notin f^{-1}X & (f \uparrow X)(x^-) &= (\iota^- \circ F)(x) \\(f \uparrow X)(\vec{x}) &= f(x) \uparrow X & (f \uparrow X)(x^+) &= (\iota^+ \circ F)(x)\end{aligned}$$

For a cell $c \in C_m$ with $\text{supp}(c) \cap X \neq \emptyset$, we define $c \uparrow X$ by recursion:

$$\begin{aligned}v \uparrow X &= \vec{v} \\ \text{coh}_{B, u \rightarrow v}(f) \uparrow X &= (f \uparrow X)(\text{coh}_{B, u \rightarrow v}(\text{id}) \uparrow X_f)\end{aligned}$$

To close the recursion, we need to define the naturality of operations.

Limitation

We have only defined $\text{coh}_{B,u \rightarrow v}(\text{id}) \uparrow X$ when the depth of X is at most 1, where

$$\text{depth}_X(c) = \max\{\dim c - \dim x : x \in X \cap \text{supp}(c)\}$$

$$\text{depth}_X(C) = \max\{\text{depth}_X(v) : v \in \coprod V_n^C\}$$

- In the case $\text{depth}_X(B) = 0$, we have that $B \uparrow X$ is again a pasting context and we may let

$$\text{coh}_{B,u \rightarrow v}(\text{id}) \uparrow X = \text{coh}_{B \uparrow X, \iota^-(c) \curvearrowright \iota^+(c)}(\text{id})$$

- The case $\text{depth}_X(c) = \text{depth}_X(B) = 1$ is more technical.

Naturality of a coherence

Consider the following pasting diagram and cell over

$$B = \bullet^x \xrightarrow{f} \bullet^y \xrightarrow{g} \bullet^z \xrightarrow{h} \bullet^w \quad c = \alpha_{f,g,h}: (h \circ g) \circ f \rightarrow h \circ (g \circ f)$$

Its naturality with respect to $X = \{f, g, h\}$ is an operation given by:

$$B \uparrow X = \bullet^x \begin{array}{c} \xrightarrow{f^-} \\ \Downarrow \vec{f} \\ \xrightarrow{f^+} \end{array} \bullet^y \begin{array}{c} \xrightarrow{g^-} \\ \Downarrow \vec{g} \\ \xrightarrow{g^+} \end{array} \bullet^z \begin{array}{c} \xrightarrow{h^-} \\ \Downarrow \vec{h} \\ \xrightarrow{h^+} \end{array} \bullet^w \quad c \uparrow X = \begin{array}{ccc} (h^- \circ g^-) \circ f^- & \xrightarrow{\alpha_{f^-, g^-, h^-}} & h^- \circ (g^- \circ f^-) \\ (\vec{h} \circ \vec{g}) \circ \vec{f} \downarrow & \xleftarrow{\alpha_{f,g,h} \uparrow X} & \downarrow \vec{h} \circ (\vec{g} \circ \vec{f}) \\ (h^+ \circ g^+) \circ f^+ & \xrightarrow{\alpha_{f^+, g^+, h^+}} & h^+ \circ (g^+ \circ f^+) \end{array}$$

Naturality of a composite

Consider the following pasting diagram and the whiskering

$$B = x \begin{array}{c} \xrightarrow{f} \\ \Downarrow a \\ \xrightarrow{g} \end{array} y \xrightarrow{h} z \qquad c = h \circ a : x \rightarrow z$$

Its naturality with respect to $\{h\}$ is an interchanger:

$$B \uparrow h = x \begin{array}{c} \xrightarrow{f} \\ \Downarrow a \\ \xrightarrow{g} \end{array} y \begin{array}{c} \xrightarrow{h^-} \\ \Downarrow \vec{h} \\ \xrightarrow{h^+} \end{array} z \qquad c \uparrow X : (\vec{h} \circ_0 g) \circ_1 (h^- \circ_0 a) \rightarrow (a \circ h^+) \circ (\vec{h} \circ_0 f)$$

Naturality of a composite

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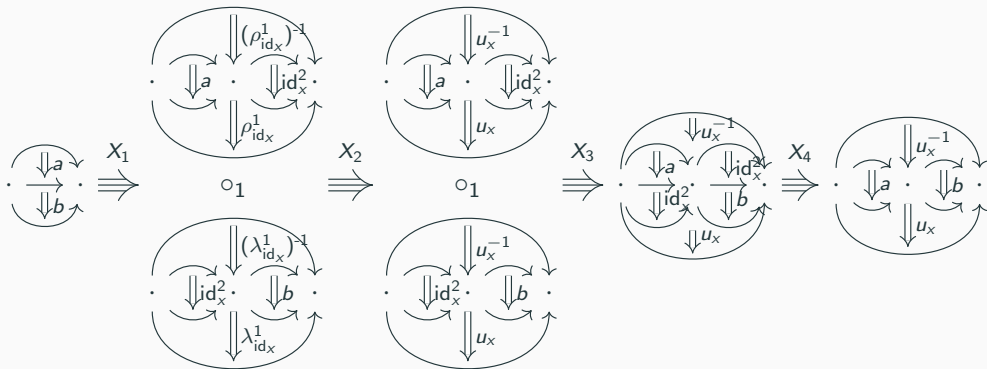
Its naturality with respect to $\{x, f, y, g, z\}$ is a composite of 1-cylinders:

$$B \uparrow h = \begin{array}{ccccc} x^- & \xrightarrow{f^-} & y^- & \xrightarrow{g^-} & z^- \\ \downarrow \vec{x} & \swarrow \vec{f} & \downarrow \vec{y} & \swarrow \vec{g} & \downarrow \vec{z} \\ x^+ & \xrightarrow{f^+} & y^+ & \xrightarrow{g^+} & z^+ \end{array}$$

$$\begin{aligned} c \uparrow X : \vec{z} \circ (g^- \circ f^-) & \\ \longrightarrow (\vec{z} \circ g^-) \circ f^- & \\ \longrightarrow (g^+ \circ \vec{y}) \circ f^- & \\ \longrightarrow g^+ \circ (\vec{y} \circ f^-) & \\ \longrightarrow g^+ \circ (f^+ \circ \vec{x}) & \\ \longrightarrow (g^+ \circ f^+) \circ \vec{x} & \end{aligned}$$

A consequence: The Eckmann Hilton argument

- Consider the computad C with one object $(x : \star)$ and two 2-cells $(a, b : 1_x \rightarrow 1_x)$.
- The Eckmann Hilton argument is a cell $(e : b \circ_1 a \Rightarrow a \circ_1 b)$ showing that composition is weakly commutative.



Thank you for your attention!