Coinductive invertibility in higher categories

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- Strict n-categories naturally form an $(n+1)$-category.
- Manifolds with cobordisms assemble into (∞, n)-categories.

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Programme

Our goal is to develop an algebraic (set-based) structure modelling (∞, ∞)-categories. Our candidate are the ω-categories of Batanin [3] and Leinster [13].

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- A morphism $f\colon x \to y$ in an $(n+1)$-category is invertible if there exist $f^l, f^r\colon y \to x$ and invertible
$$\epsilon_f\colon f \circ f^l \Rightarrow 1(x) \qquad\qquad \eta_f\colon 1(y) \Rightarrow f^r \circ f$$
in the n-categories $\mathrm{Hom}(x,x)$ and $\mathrm{Hom}(y,y)$.

$$\begin{tikzcd}
y \arrow[rr, equal] \arrow[dr, "f^l"'] & {} \arrow[d, phantom, "\wr\Uparrow\epsilon_f"] & y \arrow[dr, "f^r"] & \\
& x \arrow[ur, "f" description] \arrow[rr, equal] & {} \arrow[u, phantom, "\wr\Uparrow\eta_f"] & x
\end{tikzcd}$$

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& x \arrow[ur, "f" description] \arrow[rr, equal] & {} & x
\end{tikzcd}$$

Definition

For ω-categories this definition is made coinductive: invertibility of a morphism is witnessed by an infinite tower of invertible higher morphisms.

Objectives of this talk

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2. What can we freely generate ω-categories from?

3. Can we make some of the generating data as invertible?

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2. What can we freely generate ω-categories from?
   - Globular sets is an easy answer, but not the most general one
   - Batanin [2] and Burroni [8] use computads (a.k.a. polygraphs)
   - Dean, Finster, M., Reutter and Vicary [9] introduce an inductive presentation ω-categories freely generated from a computad
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Remark

All definitions of ω-categories have been shown equivalent by Ara [1], Bourke [7], Benjamin, Finster, Mimram [6], and Benjamin, M., Sarti [5].

Strict higher categories i

Definition

Strict 0-categories are sets.
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Pros

- They admit a simple inductive definition.
- Their homotopy theory is understood [12].

Cons

- Most examples of higher categories are not strict.
- Not all tricategories are equivalent to strict ones.

Globular sets

Definition

The category $\mathbb{G}$ of globes is the category generated by the following graph subject to the globularity relations:

$$0 \underset{\tau_0}{\overset{\sigma_0}{\rightrightarrows}} 1 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 2 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \dots$$

$$\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$$
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$$\begin{tikzcd} 0 \arrow[r, shift left, "\sigma_0"] \arrow[r, shift right, "\tau_0"'] & 1 \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\tau_1"'] & 2 \arrow[r, shift left, "\sigma_2"] \arrow[r, shift right, "\tau_2"'] & \dots \end{tikzcd}$$

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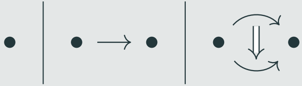
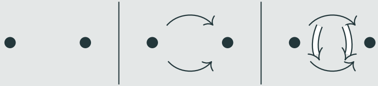
Example

The disks $\mathbb{D}^n = \mathbb{G}(-, n)$ are the representable globular sets.



Example

The spheres $\mathbb{S}^n = \mathbb{D}^{n+1} \setminus \{\mathrm{id}_{n+1}\}$ are their boundaries.



Strict higher categories ii

Definition

A strict ω-category is a globular set X equipped with identity and composition operations for $k < n$:

$$\circ_k \colon X_n \times_{X_k} X_n \to X_n \qquad\qquad 1_n \colon X_n \to X_{n+1}$$

satisfying associativity, unitality and interchange laws.

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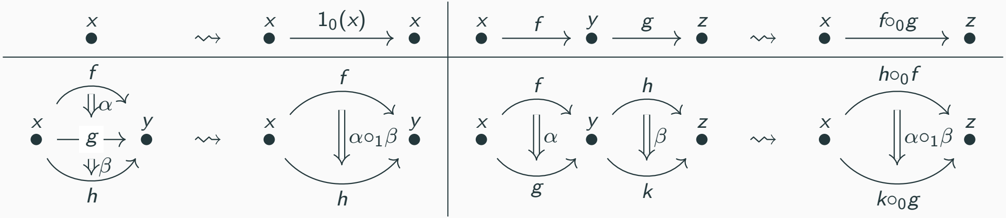

Pasting diagrams

Pasting diagrams are a family of globular sets that admit a unique composite in any strict ω-category. They are generated inductively from $\mathbb{D}^0$ using the suspension and wedge sum operations:

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Definition

The suspension of a globular set X has two objects u, v with $\mathrm{Hom}(u, v) = X$ and all other homs empty.

$$x \xrightarrow{f} y \xrightarrow{g} z \quad \rightsquigarrow \quad \begin{tikzcd} u \arrow[d, bend right=80, "x"'] \arrow[d, "y" near end] \arrow[d, bend left=80, "z"] \arrow[d, phantom, shift right=4, "\overset{f}{\Rightarrow}"] \arrow[d, phantom, shift left=4, "\overset{g}{\Rightarrow}"] \\ v \end{tikzcd}$$

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$$x \xrightarrow{f} y \xrightarrow{g} z \quad \rightsquigarrow \quad \begin{tikzcd} u \arrow[d, bend right=60, "x"'] \arrow[d, "y"] \arrow[d, bend left=60, "z"] \\ v \end{tikzcd} \quad f: x \Rightarrow y,\ g: y \Rightarrow z$$

Definition

The wedge sum of globular sets X and Y with chosen objects $x, x' \in X_0$ and $y, y' \in Y_0$ is the pushout:

$$\begin{tikzcd} & & X \vee Y & & \\ & X \arrow[ur, dashed] & & Y \arrow[ul, dashed] & \\ \mathbb{D}^0 \arrow[ur, "x"] & & \mathbb{D}^0 \arrow[ul, "x'"'] \arrow[ur, "y"] & & \mathbb{D}^0 \arrow[ul, "y'"'] \end{tikzcd}$$

Batanin trees

Batanin [3] showed that pasting diagrams are parametrised by rooted, planar trees:

Definition

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$$\mathrm{Pos}(\mathrm{br}[B_1, \dots, B_n]) = \bigvee_{i=1}^{n} \Sigma \, \mathrm{Pos}(B_i)$$

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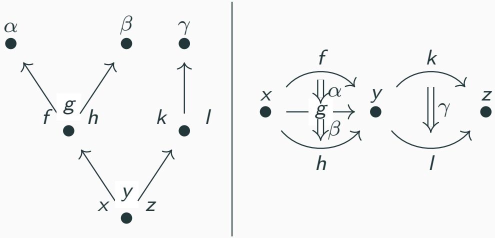

$$B = \mathrm{br}[\mathrm{br}[\mathrm{br}[], \mathrm{br}[]], \mathrm{br}[\mathrm{br}[]]]$$

Towards weak higher categories

Guiding principles

To define weak ω-categories, we replace uniqueness with uniqueness up to equivalence:

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- Parallel ways to compose the source and target of a pasting diagram give rise to a way to compose the entire pasting diagram.
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$$\begin{tikzcd} \overset{x}{\bullet} \arrow[r, "f"] & \overset{y}{\bullet} \arrow[r, "g"] & \overset{z}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \overset{x}{\bullet} \arrow[r, "f \circ g"] & \overset{y}{\bullet} \end{tikzcd}$$

$$\begin{tikzcd} \overset{x}{\bullet} \arrow[r, "f"] & \overset{y}{\bullet} \arrow[r, "g"] & \overset{z}{\bullet} \arrow[r, "h"] & \overset{w}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \overset{x}{\bullet} \arrow[r, bend left, "(f \circ g) \circ h"] \arrow[r, bend right, "f \circ (g \circ h)"'] \arrow[r, phantom, "\Downarrow \alpha_{f,g,h}"] & \overset{w}{\bullet} \end{tikzcd}$$

$$\overset{x}{\bullet} \quad \rightsquigarrow \quad \begin{tikzcd} \overset{x}{\bullet} \arrow[r, "1(x)"] & \overset{x}{\bullet} \end{tikzcd}$$

$$\begin{tikzcd} \overset{x}{\bullet} \arrow[r, "f"] & \overset{y}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \overset{x}{\bullet} \arrow[r, bend left, "f \circ 1(y)"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow \rho_f"] & \overset{x}{\bullet} \end{tikzcd}$$

Computads for higher categories

Street [15] defined computads as ways to generate 2-categories. Burroni [8] generalised them to generate strict ω-categories. Batanin [2] further generalised them to generate algebras for finitary monads on globular sets, including strict and weak ω-categories.

Idea

An n-computad is a diagram of sets of the following form

$$\begin{tikzcd}
V_0^C \arrow[d] & V_1^C \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & V_2^C \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & \cdots \arrow[dl, shift left] \arrow[dl, shift right] & V_n^C \arrow[dl, shift left] \arrow[dl, shift right] \\
FC_0 & FC_1 \arrow[l, shift left] \arrow[l, shift right] & FC_2 \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}$$

where FC_k is the T-algebra generated by the k-computad $(V_0^C, \ldots, V_k^C)$.

Inductive framework i

We will describe the category of computads and an adjunction that induces the *free weak ω-category* monad on globular sets:

$$\begin{tikzcd} \mathsf{Glob} \arrow[r, shift left, ""] \arrow[r, phantom, "\perp"] & \mathsf{Comp} \arrow[l, shift left, "\mathsf{Cell}"] \end{tikzcd}$$

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$$
\begin{tikzcd}
\mathsf{Glob} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] & \mathsf{Comp} \arrow[l, shift left=1.5ex, "\mathsf{Cell}"]
\end{tikzcd}
$$

Our computads will have some generators marked invertible and their morphisms will send generators to arbitrary cells. This will give the desired morphism of adjunctions:

$$
\begin{tikzcd}
\mathsf{Comp} \arrow[rr, "K"] \arrow[dr, shift left, "\mathsf{Cell}"] & & \mathsf{Cat}_\omega \arrow[dl, shift left, "U"] \\
& \mathsf{Glob} \arrow[ul, shift left] \arrow[ul, phantom, "\dashv"] \arrow[ur, shift left, "F"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}
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Inductive framework ii

We will define by induction on the dimension $-1 \leq n \leq \omega$ the following:

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5. A subset $\mathsf{Full}_n(B) \subseteq \mathsf{Glob}(\mathbb{S}^n, \mathsf{Cell}_n \, \mathsf{Pos}\, B)$ for each Batanin tree B

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6. a functor $\mathsf{Mark}_n \colon \mathsf{Comp}_n \to \mathsf{Set}$
7. a natural transformations $\mathsf{fwd}, \mathsf{linv}, \mathsf{rinv} \colon \mathsf{Mark}_n \Rightarrow \mathsf{Cell}_n$

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starting from $\mathsf{Comp}_{-1} = \mathbb{1}$ and $\mathrm{Cell}_{-1} = \mathrm{Mark}_{-1} = \emptyset$.

Computads and their morphisms

Definition

An $(n+1)$-computad C with invertible generators consists of:

- an n-computad $\mathrm{tr}_n\, C$,
- two sets V^C_{n+1} and U^C_{n+1} of ordinary and invertible generators respectively,
- an attaching function $\phi^C_n \colon V^C_{n+1} + U^C_{n+1} \to \mathrm{Glob}(\mathbb{S}^n, \mathrm{Cell}_n\, \mathrm{tr}_n\, C)$

Definition

A morphism $f \colon C \to D$ of $(n+1)$-computads consists of:

- a morphism $\mathrm{tr}_n\, f \colon \mathrm{tr}_n\, C \to \mathrm{tr}_n\, D$
- a function $f_V \colon V^C_{n+1} \to \mathrm{Cell}_{n+1}(D)$
- a function $f_U \colon U^C_{n+1} \to \mathrm{Mark}_{n+1}(D)$

preserving the source and target of each ordinary and invertible generator.

Cells and their boundaries

Definition

The set $\mathrm{Mark}_{n+1}(C)$ is inductively generated by:

- $\mathrm{igen}(u)$ for $u \in U^C_{n+1}$
- $\mathrm{lunit}(i)$ and $\mathrm{runit}(i)$ for $i \in \mathrm{Mark}_n(\mathrm{tr}_n\, C)$

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The set $\mathrm{Cell}_{n+1}(C)$ is inductively generated by:

- $\mathrm{gen}(v)$ for $v \in V^C_{n+1}$
- $\mathrm{fwd}(i)$, $\mathrm{linv}(i)$ and $\mathrm{rinv}(i)$ for $i \in \mathrm{Mark}_{n+1}(C)$
- $\mathrm{coh}_{B,A}[f]$ for
  - Batanin tree B with $\dim B \leq n+1$
  - $A \in \mathrm{Full}_n(B)$
  - $f\colon \mathrm{Pos}(B) \to C$

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  - $f\colon\ \mathsf{Pos}(B) \to C$

$$\begin{aligned}
\mathsf{ibdry}\colon\ \mathsf{Mark}_{n+1}(C) &\to \mathsf{Glob}(\mathbb{S}^n, \mathsf{Cell}_n\, \mathsf{tr}_n\, C)\\
\mathsf{igen}(u) &\mapsto \phi^C_{n+1}(u)\\
\mathsf{lunit}(i) &\mapsto (\mathsf{linv}(i) \circ \mathsf{fwd}(i), 1)\\
\mathsf{runit}(i) &\mapsto (1, \mathsf{fwd}(i) \circ \mathsf{rinv}(i))
\end{aligned}$$

$$\begin{aligned}
\mathsf{bdry}\colon\ \mathsf{Cell}_{n+1}(C) &\to \mathsf{Glob}(\mathbb{S}^n, \mathsf{Cell}_n\, \mathsf{tr}_n\, C)\\
\mathsf{gen}(v) &\mapsto \phi^C_{n+1}(v)\\
\mathsf{fwd}(i) &\mapsto \mathsf{ibdry}(i)\\
\mathsf{linv}(i) &\mapsto \mathsf{ibdry}(i)^{\mathsf{op}}\\
\mathsf{rinv}(i) &\mapsto \mathsf{ibdry}(i)^{\mathsf{op}}\\
\mathsf{coh}_{B,A}[f] &\mapsto (\mathsf{tr}_n\, f)_*(A)
\end{aligned}$$

Full spheres

Full spheres are an analogue of the admissible pairs of maps in the sense of Maltsiniotis [14]. They are the boundaries of the operations of ω-categories:

Definition

The support of a cell $c \in \mathrm{Cell}_n(C)$ is the set of generators that appear in c or its boundary. A sphere $A = (a, b)$ of $\mathrm{Cptd}_{n+1}\,\mathrm{Pos}(B)$ is full when

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$$B_2 = \overset{x}{\bullet} \xrightarrow{f} \overset{y}{\bullet} \xrightarrow{g} \overset{z}{\bullet} \xrightarrow{h} \overset{w}{\bullet}$$

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A_3 &= \quad (x, w) \quad \text{full} \\
A_4 &= \quad (f \circ g \circ h, (f \circ g) \circ h) \quad \text{full} \\
A_5 &= \quad ((f \circ g) \circ h, f \circ g \circ h) \quad \text{full}
\end{aligned}$$

$$\begin{aligned}
f \circ g &= \mathrm{coh}_{B_1, A_1}[f \mapsto f, g \mapsto g] \\
(f \circ g) \circ h &= \mathrm{coh}_{B_1, A_1}[f \mapsto f \circ g, g \mapsto h] \\
f \circ g \circ h &= \mathrm{coh}_{B_2, A_3}[\mathrm{id}]
\end{aligned}$$

Computads and ω-categories

Definition

The category Comp of computads with invertible generators is the limit:

$$\dots \xrightarrow{\mathsf{tr}_n} \mathsf{Comp}_n \xrightarrow{\mathsf{tr}_{n-1}} \dots \xrightarrow{\mathsf{tr}_1} \mathsf{Comp}_1 \xrightarrow{\mathsf{tr}_0} \mathsf{Comp}_0$$

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The remaining data gives rise to an adjunction:

$$\begin{tikzcd} \mathsf{Glob} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] & \mathsf{Comp} \arrow[l, shift left=1.5ex, "\mathsf{Cell}"] \end{tikzcd}$$

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Theorem

The category of weak ω-categories is the category of T-algebras.

$$\begin{tikzcd} \mathrm{Comp} \arrow[rr, dashed, "K"] \arrow[dr, shift left, "\mathrm{Cell}"] & & \mathrm{Cat}_\omega \arrow[dl, shift left, "U"] \\ & \mathrm{Glob} \arrow[ul, shift left] \arrow[ul, phantom, "\perp"] \arrow[ur, shift left, "F"] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

Variants of computads

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An ordinary computad is a computad with no invertible generators.

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$$\begin{tikzcd} \mathrm{Comp}^{\mathrm{ord},\mathrm{rig}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Comp}^{\mathrm{ord}} \arrow[d, hook] \\ \mathrm{Comp}^{\mathrm{rig}} \arrow[r] & \mathrm{Comp} \end{tikzcd}$$

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Theorem

The rigid subcategories are presheaf topoi.

Remark

This fails in the strict case!

Weak equivalences and trivial fibrations

Definition

A weak equivalence $f\colon X \to Y$ between ω-categories is a morphism satisfying that:

- for every $y \in Y_0$ there exists $x \in X_0$ and and $f(x) \xrightarrow{\sim} y$
- for every $x, x' \in X_0$ and $y\colon f(x) \to f(x')$, there exists $g\colon x \to x'$ and $f(g) \xrightarrow{\sim} y$

$$\begin{tikzcd}
x \arrow[r, dashed, "g"] \arrow[d, mapsto] & x' \arrow[d, mapsto] \\
f(x) \arrow[r, dashed, bend left, "f(g)"{name=U}] \arrow[r, bend right, "y"'{name=D}] & f(x') \arrow[from=U, to=D, Rightarrow, dashed, "\wr"]
\end{tikzcd}$$

$$\begin{tikzcd}
F^T\mathbb{S}^n \arrow[r, "{(x,x')}"] \arrow[d, hook] & X \arrow[d, "f"] \\
F^T\mathbb{D}^{n+1} \arrow[r, "y"'] \arrow[ur, dashed, "g", "\sim"'] & Y
\end{tikzcd}$$

Definition

A trivial fibration is a morphism that has the right lifting property with respect to $F^T\mathbb{S}^n \hookrightarrow F^T\mathbb{D}^{n+1}$ for all $n \geq -1$.

Every ω-category is free!

Theorem

Every ω-category is weakly equivalent to one generated by a computad.

Idea of proof.

- The inclusion $\mathsf{Comp}^{\mathrm{ord,rig}} \hookrightarrow \mathsf{Comp} \xrightarrow{K} \mathsf{Cat}_\omega$ admits a right adjoint Q.
- The counit $KQX \to X$ is a trivial fibration and hence a weak equivalence. $\square$

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Remark

Computads are the cofibrant objects for the factorisation system generated by the sphere to disk inclusions.

Did we need invertibility?

Proposition

The inclusion $\mathsf{Comp}^{\mathsf{ord}} \hookrightarrow \mathsf{Comp}$ admits a right adjoint R that preserves the generated ω-category.

Idea of proof.

We construct the computad RC by induction on the dimension. Each element of $\mathsf{Mark}_n(C)$ gives rise to three generators of RC, one for its itself and two for its inverses. $\square$

Nonetheless, we get a very succinct description of certain weak ω-categories by allowing invertible generators. For example, we can define the walking isomorphism $\mathbb{E}^{n+1}$ to be the computad extending $\mathbb{S}^n$ with a single invertible generator.

Future directions and consequences

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Future directions and consequences

- This work is part of our pursuit of a model structure on Cat_ω.
- What remaining to be done is showing that certain cylinder objects exist.
- This is a stepping stone towards comparing (marked) ω-categories with other models of (∞, ∞)-categories, such as Θ-spaces, or complicial sets.
- Extending the theory to ω-groupoids would yield a new model for homotopy types, giving new insight on their algebraic structure and strengthening their connection to intensional type theory.

Thank you for your attention!

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Bonus: The grammar of CaTT

We can alternatively reason about weak ω-categories and coinductive invertibilty using dependent type theory. This allows us to construct proof assistants for reasoning about weak ω-categories.

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We can alternatively reason about weak ω-categories and coinductive invertibilty using dependent type theory. This allows us to construct proof assistants for reasoning about weak ω-categories. The grammar of the type theory CaTT:

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\begin{array}{llcccc}
\text{Variables} & x, y & ::= & x \in \mathbb{N} & & \\
\text{Terms} & t, u & ::= & x & \mid & \mathrm{coh}_{\Delta, A}[\sigma] \\
\text{Types} & A, B & ::= & \star & \mid & u \to_A v. \\
\text{Contexts} & \Gamma, \Delta & ::= & \emptyset & \mid & \Gamma, x \colon A \\
\text{Substitutions} & \sigma, \tau & ::= & \langle\rangle & \mid & \langle \sigma, x \mapsto t \rangle
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\end{array}$$

Its judgement forms are the usual ones for dependent type theory:

$$\Gamma \vdash \qquad \Gamma \vdash A \qquad \Gamma \vdash t \colon A \qquad \Gamma \vdash \sigma : \Delta$$

with three additional auxiliary judgement forms:

$$\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash_{\mathsf{ps}} x \colon A \qquad \Gamma \vdash_{\mathsf{full}} A$$

The rules of CaTT

The rules of CaTT contain the usual structural rules for contexts, substitutions and variables. Its type and term formation rules are the following ones:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash u : A \qquad \Gamma \vdash v : A}{\Gamma \vdash u \to_A v} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash_{\mathsf{full}} A \qquad \Delta \vdash \delta : \Gamma}{\Delta \vdash \mathsf{coh}_{\Gamma,A}[\sigma] : A[\sigma]}$$

The rules for pasting contexts and full types are more involved, but analogous to the ones described in the setting of computads.

Theorem

Contexts of CaTT are finite computads and models of CaTT are weak ω-categories.

An extension of CaTT

We may extend CaTT with a new type former of invertible terms and six destructors:

$$\frac{\Gamma \vdash f \colon x \to_A y}{\Gamma \vdash \mathsf{Inv}(f)} \qquad \frac{\Gamma \vdash i : \mathsf{Inv}_A(f)}{\begin{array}{rl} \Gamma \vdash \mathsf{linv}(i) : A^{\mathsf{op}} & \Gamma \vdash \mathsf{rinv}(i) : A^{\mathsf{op}} \\ \Gamma \vdash \mathsf{lunit}(i) : \mathsf{linv}(i) * f \to \mathsf{id} & \Gamma \vdash \mathsf{runit}(i) : \mathsf{id} \to f * \mathsf{rinv}(i) \\ \Gamma \vdash \mathsf{ilunit}(i) : \mathsf{Inv}(\mathsf{lunit}(i)) & \Gamma \vdash \mathsf{irunit}(i) : \mathsf{Inv}(\mathsf{runit}(i)) \end{array}}$$

We also add a constructor and a recursion principle as well as β and η rules.

Semantics

$$
\begin{tikzcd}
S_{CaTT} \arrow[r, hook] \arrow[d, hook] & S_{ICaTT} \arrow[d, hook] \\
\mathrm{Mod}(CaTT) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Mod}(ICaTT) \arrow[l, shift left]
\end{tikzcd}
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The extension of CaTT with invertible terms is conservative.

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Via the adjunction on the left we can describe some infinite computads as contexts of the extended type theory.

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Theorem

We have semantics in fibrant marked weak ω-categories.