Invertibility in higher categories

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The need for higher categories

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   Rings with bimodules form a bicategory or a double category.
   Categories with profunctors form a bicategory.

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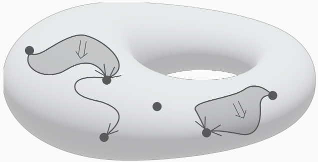
3. Each topological space X forms an ∞-groupoid $\Pi_\infty(X)$ with:
   - objects the points of X,
   - 1-morphisms the paths in X,
   - 2-morphisms the path homotopies,
   - higher morphisms the homotopies between homotopies.



Variants of higher structures

Models for higher categories are roughly classified by:

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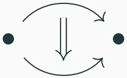
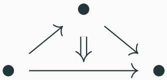
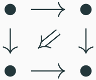


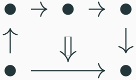
Globular



Simplicial



Cubical



Opetopic

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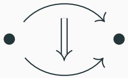

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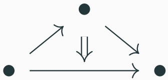

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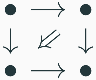

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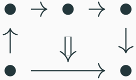

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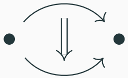

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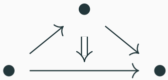

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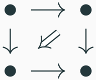

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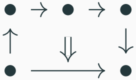

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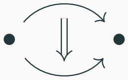

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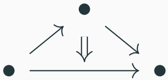

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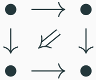

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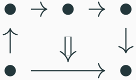

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Remark

We will focus on algebraic globular weak (∞, ∞)-categories, henceforth ω-categories.

Invertibility in higher categories

Invertible morphisms allow us to define equivalence within and between higher categories.

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- A morphism $f\colon x \to y$ in an $(n+1)$-category is invertible if there exist $f^l, f^r\colon y \to x$ and invertible
$$\epsilon_f\colon f \circ f^l \Rightarrow 1(x) \qquad\qquad \eta_f\colon 1(y) \Rightarrow f^r \circ f$$
in the n-categories $\mathrm{Hom}(x,x)$ and $\mathrm{Hom}(y,y)$.

$$\begin{tikzcd}
y \arrow[rr, equal] \arrow[dr, "f^l"'] & & y \arrow[dr, "f^r"] \arrow[d, phantom, "\wr\Uparrow\eta_f"] & \\
& x \arrow[ur, "f" description] \arrow[rr, equal] \arrow[u, phantom, "\wr\Uparrow\epsilon_f"] & {} & x
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Definition

For ω-categories this definition is made coinductive: invertibility of a morphism is witnessed by an infinite tower of invertible higher morphisms.

Objectives of this talk

1. What is an ω-category?

2. What can we freely generate ω-categories from?

3. Can we make some of the generating morphisms as invertible?

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   - Batanin [3] and Leinster [13] define them as using globular operads
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   - Globular sets is an easy answer, but not the most general one
   - Batanin [2] and Burroni [8] use computads (a.k.a. polygraphs)
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Remark

All definitions of ω-categories have been shown equivalent by Ara [1], Bourke [7], Benjamin, Finster, Mimram [6], and Benjamin, M., Sarti [5].

Strict ω-categories i

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Pros

- They admit a simple inductive definition.
- Their homotopy theory is understood [12].

Cons

- Most examples of higher categories are not strict.
- Not all tricategories are equivalent to strict ones.

Globular sets

Definition

The category $\mathbb{G}$ of globes is the category generated by the following graph subject to the globularity relations:

$$0 \overset{\sigma_0}{\underset{\tau_0}{\rightrightarrows}} 1 \overset{\sigma_1}{\underset{\tau_1}{\rightrightarrows}} 2 \overset{\sigma_2}{\underset{\tau_2}{\rightrightarrows}} \dots \qquad \begin{aligned} \sigma_{n+1} \circ \sigma_n &= \tau_{n+1} \circ \sigma_n \\ \sigma_{n+1} \circ \tau_n &= \tau_{n+1} \circ \tau_n \end{aligned}$$

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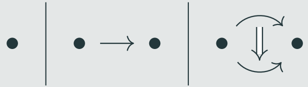
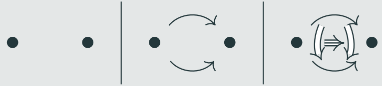
Example

The disks $\mathbb{D}^n = \mathbb{G}(-, n)$ are the representable globular sets.



Example

The spheres $\mathbb{S}^n = \mathbb{D}^{n+1} \setminus \{\mathrm{id}_{n+1}\}$ are their boundaries.



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A strict ω-category is a globular set X equipped with identity and composition operations for $k < n$:

$$\circ_k \colon X_n \times_{X_k} X_n \to X_n \qquad\qquad 1_n \colon X_n \to X_{n+1}$$

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$$\overset{x}{\bullet} \quad \rightsquigarrow \quad \begin{tikzcd} \overset{x}{\bullet} \arrow[r, "1_0(x)"] & \overset{x}{\bullet} \end{tikzcd}$$

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$$\begin{tikzcd} \overset{x}{\bullet} \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \overset{y}{\bullet} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \overset{x}{\bullet} \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Downarrow\alpha \circ_1 \beta"] & \overset{y}{\bullet} \end{tikzcd}$$

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Pasting diagrams

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The suspension of a globular set X has two objects u, v with $\mathrm{Hom}(u, v) = X$ and all other homs empty.

$$x \xrightarrow{f} y \xrightarrow{g} z \quad \rightsquigarrow \quad \begin{tikzcd} u \arrow[d, bend right=80, "x"'{name=X}] \arrow[d, "y"{name=Y}] \arrow[d, bend left=80, "z"{name=Z}] \arrow[from=X, to=Y, Rightarrow, "f"] \arrow[from=Y, to=Z, Rightarrow, "g"] \\ v \end{tikzcd}$$

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Definition

The wedge sum of globular sets X and Y with chosen objects $x, x' \in X_0$ and $y, y' \in Y_0$ is the pushout:

$$\begin{tikzcd} & & X \vee Y \arrow[dd, phantom, "\vee" very near start] & & \\ & X \arrow[ur, dashed] & & Y \arrow[ul, dashed] & \\ \mathbb{D}^0 \arrow[ur, "x"] & & \mathbb{D}^0 \arrow[ul, "x'"'] \arrow[ur, "y"] & & \mathbb{D}^0 \arrow[ul, "y'"'] \end{tikzcd}$$

Batanin trees

Batanin [3] showed that pasting diagrams are parametrised by rooted, planar trees:

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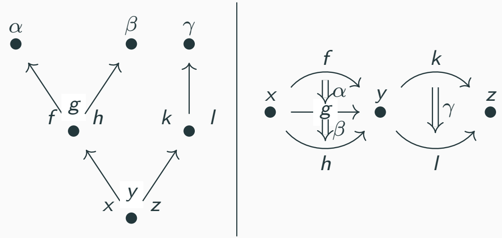

$$B = \mathrm{br}[\mathrm{br}[\mathrm{br}[], \mathrm{br}[]], \mathrm{br}[\mathrm{br}[]]]$$

Towards weak ω-categories

Guiding principles

To define weak ω-categories, we replace uniqueness with uniqueness up to equivalence:

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Guiding principles

To define weak ω-categories, we replace uniqueness with uniqueness up to equivalence:

- Parallel ways to compose the source and target of a pasting diagram give rise to a way to compose the entire pasting diagram.
- Parallel composites or coherences over a pasting diagram are related by a coherence operation over the same pasting diagram.

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Strict ω-categories iii

Definition

Strict ω-categories are algebras for a monad T^{str} on Glob:

$$(T^{\mathrm{str}}X)_n = \coprod_{\dim B \leq n} \mathrm{Glob}(\mathrm{Pos}(B), X)$$

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It follows that there is an adjunction:

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Question

What about more general shapes of generating data:

- simplices, cubes, opetopes...
- the walking isomorphism $x \xrightarrow{\sim} y$
- the walking commmutative square

Street [15] defined computads as ways to generate 2-categories. Burroni [8] generalised them to generate strict ω-categories. Batanin [2] further generalised them to generate algebras for finitary monads on globular sets, including strict and weak ω-categories.

Idea

An n-computad is a diagram of sets of the following form

$$\begin{tikzcd}
V_0^C \arrow[d] & V_1^C \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & V_2^C \arrow[d] \arrow[dl, shift left] \arrow[dl, shift right] & \dots \arrow[dl, shift left] \arrow[dl, shift right] & V_n^C \arrow[dl, shift left] \arrow[dl, shift right] \\
TC_0 & TC_1 \arrow[l, shift left] \arrow[l, shift right] & TC_2 \arrow[l, shift left] \arrow[l, shift right] & \dots \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}$$

where TC_k is the T-algebra generated by the k-computad $(V_0^C, \dots, V_k^C)$.

The formal specification

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To that end we define mutually recursively on the dimension n:

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starting from $\mathsf{Comp}_{-1} = \mathbb{1}$, $\mathsf{Sphere}_{-1} = \{\star\}$ and $\mathsf{Cell}_{-1} = \mathsf{Mark}_{-1} = \emptyset$.

Computads and their morphisms

Definition

An $(n+1)$-computad C with invertible generators consists of:

- an n-computad $\operatorname{tr}_n C$
- two sets V^C_{n+1} and U^C_{n+1} of ordinary and invertible generators
- a function $\phi^C_n \colon V^C_{n+1} + U^C_{n+1} \to \mathrm{Sphere}_n(C_n)$ giving the boundary of each generator

Definition

A morphism $f \colon C \to D$ of $(n+1)$-computads consists of:

- a morphism $\operatorname{tr}_n f \colon \operatorname{tr}_n C \to \operatorname{tr}_n D$
- a function $f_V \colon V^C_{n+1} \to \mathrm{Cell}_{n+1}(D)$
- a function $f_U \colon U^C_{n+1} \to \mathrm{Mark}_{n+1}(D)$

$$\begin{tikzcd} V^C_{n+1} + U^C_{n+1} \arrow[r, "\langle f_V, \mathrm{fwd} \circ f_U \rangle"] \arrow[d, "\phi^C_{n+1}"] & \mathrm{Cell}_{n+1}(D) \arrow[d, "\mathrm{bdry}"] \\ \mathrm{Sphere}_n(\operatorname{tr}_n C) \arrow[r, "\mathrm{Sphere}_n(\operatorname{tr}_n f)"] & \mathrm{Sphere}_n(\operatorname{tr}_n D) \end{tikzcd}$$

Cells and their boundaries

Definition

The set $\mathrm{Mark}_{n+1}(C)$ is inductively generated by:

- $\mathrm{igen}(u)$ for $u \in U^C_{n+1}$
- $\mathrm{lunit}(i)$ and $\mathrm{runit}(i)$ for $i \in \mathrm{Mark}_n(C)$

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The set $\mathrm{Cell}_{n+1}(C)$ is inductively generated by:

- $\mathrm{gen}(v)$ for $v \in V^C_{n+1}$
- $\mathrm{fwd}(i)$, $\mathrm{linv}(i)$ and $\mathrm{rinv}(i)$ for $i \in \mathrm{Mark}_n(C)$
- $\mathrm{coh}_{B,A}[f]$ for
    - Batanin tree B with $\dim B \leq n+1$
    - $A \in \mathrm{Full}_n(B)$
    - $f\colon \mathrm{Cptd}_{n+1}(\mathrm{Pos}(B)) \to C$

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$$
\begin{aligned}
\mathrm{ibdry}\colon\ \mathrm{Mark}_{n+1}(C) &\to \mathrm{Sphere}_n(\mathrm{tr}_n\, C) \\
\mathrm{igen}(u) &\mapsto \phi^C_{n+1}(u) \\
\mathrm{lunit}(i) &\mapsto (\mathrm{linv}(i) \circ \mathrm{fwd}(i), 1) \\
\mathrm{runit}(i) &\mapsto (1, \mathrm{fwd}(i) \circ \mathrm{rinv}(i))
\end{aligned}
$$

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\begin{aligned}
\mathrm{bdry}\colon\ \mathrm{Cell}_{n+1}(C) &\to \mathrm{Sphere}_n(\mathrm{tr}_n\, C) \\
\mathrm{gen}(v) &\mapsto \phi^C_{n+1}(v) \\
\mathrm{fwd}(i) &\mapsto \mathrm{ibdry}(i) \\
\mathrm{linv}(i) &\mapsto \mathrm{ibdry}(i)^{\mathrm{op}} \\
\mathrm{rinv}(i) &\mapsto \mathrm{ibdry}(i)^{\mathrm{op}} \\
\mathrm{coh}_{B,A}[f] &\mapsto \mathrm{Sphere}_n(\mathrm{tr}_n\, f)(A)
\end{aligned}
$$

Spheres and globular sets

Spheres are pairs of cells with the same source and target:

Definition

The set $\mathrm{Sphere}_{n+1}(C)$ and the projections natural transformations are defined by the following pullback square:

$$\begin{tikzcd}
\mathrm{Sphere}_{n+1}(C) \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cell}_{n+1}(C) \arrow[d, "\mathrm{bdry}"] \\
\mathrm{Cell}_{n+1}(C) \arrow[r, "\mathrm{bdry}"'] & \mathrm{Sphere}_n(\mathrm{tr}_n\, C)
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\mathsf{Cell}_{n+1}(C) \arrow[r, "\mathsf{bdry}"'] & \mathsf{Sphere}_n(\mathsf{tr}_n\, C)
\end{tikzcd}$$

Globular sets are the computads where the source and target of a generator are again generators:

Definition

The computad $\mathsf{Cptd}_{n+1}\, X$ for a globular set X consists of:

- the n-computad $\mathsf{Cptd}_n\, X$
- $V_{n+1} = X_{n+1}$
- $U_{n+1} = \emptyset$
- $\phi_{n+1}(x) = (\mathsf{gen}\,\mathsf{src}\,x, \mathsf{gen}\,\mathsf{tgt}\,x)$

Full spheres

Full spheres are an analogue of the admissible pairs of maps in the sense of Maltsiniotis [14]. They are the boundaries of the operations of ω-categories:

Definition

The support of a cell $c \in \mathrm{Cell}_n(C)$ is the set of generators that appear in c or its boundary. A sphere $A = (a, b)$ of $\mathrm{Cptd}_{n+1}\,\mathrm{Pos}(B)$ is full when

- the support of a contains all positions of $\partial_n^- B$
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$$B_2 = \overset{x}{\bullet} \xrightarrow{f} \overset{y}{\bullet} \xrightarrow{g} \overset{z}{\bullet} \xrightarrow{h} \overset{w}{\bullet}$$

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$$A_4 = (f \circ g \circ h, (f \circ g) \circ h) \quad \text{full}$$

$$A_5 = ((f \circ g) \circ h, f \circ g \circ h) \quad \text{full}$$

$$\begin{aligned} f \circ g &= \mathrm{coh}_{B_1, A_1}[f \mapsto f, g \mapsto g] \\ (f \circ g) \circ h &= \mathrm{coh}_{B_1, A_1}[f \mapsto f \circ g, g \mapsto h] \\ f \circ g \circ h &= \mathrm{coh}_{B_2, A_3}[\mathrm{id}] \end{aligned}$$

Computads and ω-categories

Definition

The category Comp of computads with invertible generators is the limit:

$$\dots \xrightarrow{\mathrm{tr}_n} \mathrm{Comp}_n \xrightarrow{\mathrm{tr}_{n-1}} \dots \xrightarrow{\mathrm{tr}_1} \mathrm{Comp}_1 \xrightarrow{\mathrm{tr}_0} \mathrm{Comp}_0$$

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The remaining data gives rise to an adjunction:

$$\begin{tikzcd} \mathrm{Glob} \arrow[r, shift left=2, "\mathrm{Cptd}"] \arrow[r, phantom, "\perp"] & \mathrm{Comp} \arrow[l, shift left=2, "\mathrm{Cell}"] \end{tikzcd}$$

and hence a monad T on globular sets.

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Theorem

The category of weak ω-categories is the category of T-algebras.

$$\begin{tikzcd} \mathrm{Comp} \arrow[rr, dashed, "K"] \arrow[dr, shift left, "\mathrm{Cell}"] & & \mathrm{Cat}_\omega \arrow[dl, shift left, "U^T"] \\ & \mathrm{Glob} \arrow[ul, shift left, "\mathrm{Cptd}"] \arrow[ur, shift left, "F^T"] & \end{tikzcd}$$

Variants of computads

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$$\begin{tikzcd}
\mathrm{Comp}^{\mathrm{ord},\mathrm{rig}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Comp}^{\mathrm{ord}} \arrow[d, hook] \\
\mathrm{Comp}^{\mathrm{rig}} \arrow[r] & \mathrm{Comp}
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Theorem

The rigid subcategories are presheaf topoi.

Weak equivalences and trivial fibrations

Definition

A weak equivalence $f\colon X \to Y$ between ω-categories is a morphism satisfying that:

- for every $y \in Y_0$ there exists $x \in X_0$ and and $f(x) \xrightarrow{\sim} y$
- for every $x, x' \in X_0$ and $y\colon f(x) \to f(x')$, there exists $g\colon x \to x'$ and $f(g) \xrightarrow{\sim} y$

$$\begin{tikzcd}
x \arrow[r, dashed, "g"] \arrow[d, mapsto] & x' \arrow[d, mapsto] \\
f(x) \arrow[r, dashed, bend left, "f(g)"{name=U}] \arrow[r, bend right, "y"'{name=D}] & f(x')
\arrow[from=U, to=D, Rightarrow, dashed, "\wr"]
\end{tikzcd}$$

$$\begin{tikzcd}
F^T\mathbb{S}^n \arrow[r, "{(x,x')}"] \arrow[d, hook] & X \arrow[d, "f"] \\
F^T\mathbb{D}^{n+1} \arrow[r, "y"'] \arrow[ur, dashed, "g", "\sim"'] & Y
\end{tikzcd}$$

Definition

A trivial fibration is a morphism that has the right lifting property with respect to $F^T\mathbb{S}^n \hookrightarrow F^T\mathbb{D}^{n+1}$ for all $n \geq -1$.

Every ω-category is free

Theorem

Every ω-category is weakly equivalent to one generated by a computad.

Idea of proof.

- The inclusion $\mathrm{Comp}^{\mathrm{ord,rig}} \hookrightarrow \mathrm{Comp} \xrightarrow{K} \mathrm{Cat}_\omega$ admits a right adjoint Q.
- The counit $KQX \to X$ is a trivial fibration and hence a weak equivalence. $\square$

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Remark

Computads are the cofibrant objects for the factorisation system generated by the sphere to disk inclusions.

Did we need invertibility?

Proposition

The inclusion $\mathsf{Comp}^{\mathsf{ord}} \hookrightarrow \mathsf{Comp}$ admits a right adjoint R that preserves the generated ω-category.

Idea of proof.

We construct the computad RC by induction on the dimension. Each element of $\mathsf{Mark}_n(C)$ gives rise to three generators of RC, one for its itself and two for its inverses. □

Nonetheless, we get a very succinct description of certain weak ω-categories by allowing invertible generators. For example, we can define the walking isomorphism $\mathbb{E}^{n+1}$ to be the computad extending $\mathbb{S}^n$ with a single invertible generator.

Future directions and consequences

- This work is part of our pursuit of a model structure on Cat_ω.

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- This work is part of our pursuit of a model structure on Cat_ω.
- What remaining to be done is showing that certain cylinder objects exist.
- This is a stepping stone towards comparing (marked) ω-categories with other models of (∞, ∞)-categories, such as Θ-spaces, or complicial sets.
- Extending the theory to ω-groupoids would yield a new model for homotopy types, giving new insight on their algebraic structure and strengthening their connection to intensional type theory.

Thank you for your attention!

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Bonus: The grammar of CaTT

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\begin{array}{lrcccc}
\text{Variables} & x, y & ::= & x \in \mathbb{N} & & \\
\text{Terms} & t, u & ::= & x & \mid & \mathsf{coh}_{\Delta,A}[\sigma] \\
\text{Types} & A, B & ::= & \star & \mid & u \to_A v. \\
\text{Contexts} & \Gamma, \Delta & ::= & \emptyset & \mid & \Gamma, x : A \\
\text{Substitutions} & \sigma, \tau & ::= & \langle\rangle & \mid & \langle \sigma, x \mapsto t \rangle
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$$

Its judgement forms are the usual ones for dependent type theory:

$$\Gamma \vdash \qquad\qquad \Gamma \vdash A \qquad\qquad \Gamma \vdash t \colon A \qquad\qquad \Gamma \vdash \sigma \colon \Delta$$

with three additional auxiliary judgement forms:

$$\Gamma \vdash_{\mathsf{ps}} \qquad\qquad \Gamma \vdash_{\mathsf{ps}} x \colon A \qquad\qquad \Gamma \vdash_{\mathsf{full}} A$$

The rules of CaTT

The rules of CaTT contain the usual structural rules for contexts, substitutions and variables. Its type and term formation rules are the following ones:

$$\frac{\Gamma \vdash}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash u : A \qquad \Gamma \vdash v : A}{\Gamma \vdash u \to_A v} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash_{\mathsf{full}} A \qquad \Delta \vdash \delta : \Gamma}{\Delta \vdash \mathsf{coh}_{\Gamma,A}[\sigma] : A[\sigma]}$$

The rules for pasting contexts and full types are more involved, but analogous to the ones described in the setting of computads.

Theorem

Contexts of CaTT are finite computads and models of CaTT are weak ω-categories.

An extension of CaTT

We may extend CaTT with a new type former of invertible terms and six destructors:

$$\frac{\Gamma \vdash f \colon x \to_A y}{\Gamma \vdash \mathsf{Inv}(f)} \qquad \frac{\Gamma \vdash i : \mathsf{Inv}_A(f)}{\begin{array}{rr} \Gamma \vdash \mathsf{linv}(i) : A^{\mathsf{op}} & \Gamma \vdash \mathsf{rinv}(i) : A^{\mathsf{op}} \\ \Gamma \vdash \mathsf{lunit}(i) : \mathsf{linv}(i) * f \to \mathsf{id} & \Gamma \vdash \mathsf{runit}(i) : \mathsf{id} \to f * \mathsf{rinv}(i) \\ \Gamma \vdash \mathsf{ilunit}(i) : \mathsf{Inv}(\mathsf{lunit}(i)) & \Gamma \vdash \mathsf{irunit}(i) : \mathsf{Inv}(\mathsf{runit}(i)) \end{array}}$$

We also add a constructor and a recursion principle as well as β and η rules.

Semantics

$$\begin{tikzcd}
S_{CaTT} \arrow[r, hook] \arrow[d, hook] & S_{ICaTT} \arrow[d, hook] \\
\mathrm{Mod}(CaTT) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Mod}(ICaTT) \arrow[l, shift left]
\end{tikzcd}$$

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Theorem

The extension of CaTT with invertible terms is conservative.

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Via the adjunction on the left we can describe some infinite computads as contexts of the extended type theory.

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Theorem

We have semantics in fibrant marked weak ω-categories.