

A TYPE THEORY FOR INVERTIBILITY IN WEAK ω -CATEGORIES

Thibaut Benjamin Camil Champin [Ioannis Markakis](#)

TYPES 2026

University of Cambridge

- In an 1-category, a morphism $f: x \rightarrow y$ is invertible when there exist:

$$f^{-1}: y \rightarrow x$$

$$f \circ f^{-1} = \text{id}_y$$

$$f^{-1} \circ f = \text{id}_x$$

INVERTIBILITY IN HIGHER CATEGORIES

- In an 1-category, a morphism $f: x \rightarrow y$ is invertible when there exist:

$$f^{-1}: y \rightarrow x \qquad f \circ f^{-1} = \text{id}_y \qquad f^{-1} \circ f = \text{id}_x$$

- In an arbitrary n -category, equalities are replaced by invertible cells:

$$f^{-1}: y \rightarrow x \qquad f \circ f^{-1} \xrightarrow{\sim} \text{id}_y \qquad f^{-1} \circ f \xrightarrow{\sim} \text{id}_x$$

INVERTIBILITY IN HIGHER CATEGORIES

- In an 1-category, a morphism $f: x \rightarrow y$ is invertible when there exist:

$$f^{-1}: y \rightarrow x \qquad f \circ f^{-1} = \text{id}_y \qquad f^{-1} \circ f = \text{id}_x$$

- In an arbitrary n -category, equalities are replaced by invertible cells:

$$f^{-1}: y \rightarrow x \qquad f \circ f^{-1} \xrightarrow{\sim} \text{id}_y \qquad f^{-1} \circ f \xrightarrow{\sim} \text{id}_x$$

- In an ω -category, this definition becomes coinductive: an infinite amount of data is required to witness that a cell is invertible.

WHAT IS AN ω -CATEGORY?

- An ω -category is a structure with:
 - cells of all dimensions
 - composition and identity operations
 - coherence cells witnessing associativity, unitality, interchange

¹Finster and Mimram 2017.

WHAT IS AN ω -CATEGORY?

- An ω -category is a structure with:
 - cells of all dimensions
 - composition and identity operations
 - coherence cells witnessing associativity, unitality, interchange
- They are models of the dependent type theory CaTT^1 with:
 - a base type $*$ of objects and directed identity types $u \rightarrow_A v$
 - variables and coherence terms

$$\frac{\Gamma \vdash_{\text{ps}} \quad \Gamma \vdash_{\text{full}} A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \text{coh}_{\Gamma,A}[\gamma] : A[\gamma]}$$

¹Finster and Mimram 2017.

WHAT IS AN ω -CATEGORY?

- An ω -category is a structure with:
 - cells of all dimensions
 - composition and identity operations
 - coherence cells witnessing associativity, unitality, interchange
- They are models of the dependent type theory CaTT^1 with:
 - a base type $*$ of objects and directed identity types $u \rightarrow_A v$
 - variables and coherence terms

$$\frac{\Gamma \vdash_{\text{ps}} \quad \Gamma \vdash_{\text{full}} A \quad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \text{coh}_{\Gamma, A}[\gamma] : A[\gamma]}$$

Restriction

CaTT can not reason directly about invertibility of terms.

¹Finster and Mimram 2017.

THE TYPE OF INVERTIBILITY STRUCTURES

- We extend CaTT with a new type constructor $\text{Inv}_A(t)$ witnessing invertibility of a term t .

THE TYPE OF INVERTIBILITY STRUCTURES

- We extend CaTT with a new type constructor $\text{Inv}_A(t)$ witnessing invertibility of a term t .
- It comes equipped with destructors giving for every $e : \text{Inv}_{x \rightarrow y}(f)$:

$$\begin{array}{lll} \text{linv}(e) : y \rightarrow x & \text{lunit}(e) : \text{linv}(e) \circ f \rightarrow \text{id}_y & \text{ilunit}(e) : \text{Inv}(\text{lunit}(e)) \\ \text{rinv}(e) : y \rightarrow x & \text{runit}(e) : f \circ \text{rinv}(e) \rightarrow \text{id}_x & \text{irunit}(e) : \text{Inv}(\text{runit}(e)) \end{array}$$

and with a constructor `inv` creating a witness of invertibility from such data.

THE TYPE OF INVERTIBILITY STRUCTURES

- We extend CaTT with a new type constructor $\text{Inv}_A(t)$ witnessing invertibility of a term t .
- It comes equipped with destructors giving for every $e: \text{Inv}_{x \rightarrow y}(f)$:

$$\begin{array}{lll} \text{linv}(e): y \rightarrow x & \text{lunit}(e): \text{linv}(e) \circ f \rightarrow \text{id}_y & \text{ilunit}(e): \text{Inv}(\text{lunit}(e)) \\ \text{rinv}(e): y \rightarrow x & \text{runit}(e): f \circ \text{rinv}(e) \rightarrow \text{id}_x & \text{irunit}(e): \text{Inv}(\text{runit}(e)) \end{array}$$

and with a constructor inv creating a witness of invertibility from such data.

- These constructors and destructors are related by the expected β and η rules, e.g.

$$\text{linv}(\text{inv}(f, f_l, f_r, f_{lu}, f_{ru}, f_{ilu}, f_{iru})) \equiv f_l$$

THE TYPE OF INVERTIBILITY STRUCTURES

- We extend CaTT with a new type constructor $\text{Inv}_A(t)$ witnessing invertibility of a term t .
- It comes equipped with destructors giving for every $e: \text{Inv}_{x \rightarrow y}(f)$:

$$\begin{array}{lll} \text{linv}(e): y \rightarrow x & \text{lunit}(e): \text{linv}(e) \circ f \rightarrow \text{id}_y & \text{ilunit}(e): \text{Inv}(\text{lunit}(e)) \\ \text{rinv}(e): y \rightarrow x & \text{runit}(e): f \circ \text{rinv}(e) \rightarrow \text{id}_x & \text{irunit}(e): \text{Inv}(\text{runit}(e)) \end{array}$$

and with a constructor `inv` creating a witness of invertibility from such data.

- These constructors and destructors are related by the expected β and η rules, e.g.

$$\text{linv}(\text{inv}(f, f_l, f_r, f_{lu}, f_{ru}, f_{ilu}, f_{iru})) \equiv f_l$$

- We also add a constructor `rec` allowing us to prove invertibility of some classes of cells that are closed under suspension, together with the expected β rules.

Theorem (Hoshino, Fujii, and Maehara 2024; Benjamin and Markakis 2024)

Coherence cells and composites of invertible cells are invertible.

Theorem (Hoshino, Fujii, and Maehara 2024; Benjamin and Markakis 2024)

Coherence cells and composites of invertible cells are invertible.

- This theorem suggests adding the following rule:

$$\frac{\Gamma \vdash_{\text{ps}} \quad \Gamma \vdash_{\text{full}} A \quad \Delta \vdash \gamma : \Gamma \quad \{\Delta \vdash e_x : \text{Inv}(x[\gamma])\}_{x \in \text{Var}_{\dim A+1}(\Gamma)}}{\Delta \vdash \text{can}(\Gamma, A, \gamma, \{e_x\}) : \text{Inv}(\text{coh}_{\Gamma, A}[\gamma])}$$

- We also add computation rules identifying $\text{can}(\Gamma, A, \gamma, \{e_x\})$ with the invertibility structure constructed in the theorem.

Lemma

The rewrite system of the extension ICaTT is locally confluent, but not terminating. It is normalising on objects and terms of identity types.

CONSERVATIVITY OVER CATT

Lemma

The rewrite system of the extension ICaTT is locally confluent, but not terminating. It is normalising on objects and terms of identity types.

Theorem

ICaTT is a conservative extension of CaTT. Moreover, the inclusion of syntactic categories $J: \text{CaTT} \rightarrow \text{ICaTT}$ is fully faithful.

$$\begin{array}{ccc} \text{Mod}(\text{CaTT}) & \begin{array}{c} \xrightarrow{J_!} \\ \xleftarrow{J^*} \\ \perp \end{array} & \text{Mod}(\text{ICaTT}) \\ \llbracket - \rrbracket \uparrow & & \llbracket - \rrbracket \uparrow \\ \text{CaTT} & \xrightarrow{J} & \text{ICaTT} \end{array}$$

- Terms and types of CaTT are familiarly represented by contexts \mathbb{D}^n and \mathbb{S}^{n-1} respectively.
- We define \mathbb{E}^{n+1} as the context extension of \mathbb{D}^{n+1} obtained by inverting the top-dimensional variable of \mathbb{D}^{n+1} .

²Ozornova and Rovelli 2024.

- Terms and types of CaTT are familiarly represented by contexts \mathbb{D}^n and \mathbb{S}^{n-1} respectively.
- We define \mathbb{E}^{n+1} as the context extension of \mathbb{D}^{n+1} obtained by inverting the top-dimensional variable of \mathbb{D}^{n+1} .

Theorem

The ω -category $J^[\mathbb{E}^{n+1}]$ is the walking equivalence of Ozornova and Rovelli².*

²Ozornova and Rovelli 2024.

- Terms and types of CaTT are familiarly represented by contexts \mathbb{D}^n and \mathbb{S}^{n-1} respectively.
- We define \mathbb{E}^{n+1} as the context extension of \mathbb{D}^{n+1} obtained by inverting the top-dimensional variable of \mathbb{D}^{n+1} .

Theorem

The ω -category $J^[\mathbb{E}^{n+1}]$ is the walking equivalence of Ozornova and Rovelli².*

Corollary (Fujii, Hoshino, and Maehara 2025)

Equifibrations are the morphisms with the right lifting property with respect to the source inclusion $J^[\mathbb{D}^n] \hookrightarrow J^*[\mathbb{E}^n]$*

²Ozornova and Rovelli 2024.

SEMANTICS IN MARKED ω -CATEGORIES

- Marked ω -categories are ω -categories with a specified class of marked cells closed under composition and coherences.

SEMANTICS IN MARKED ω -CATEGORIES

- Marked ω -categories are ω -categories with a specified class of marked cells closed under composition and coherences.
- We can build a semantics of ICaTT in marked ω -categories:

$$\begin{array}{ccc} \text{Mod(ICaTT)} & \xrightarrow{L} & \text{Cat}_\omega^+ \\ & \searrow J^* & \downarrow U \\ & & \text{Cat}_\omega \end{array}$$

by marking a cell if it factors through the walking equivalence.

SEMANTICS IN MARKED ω -CATEGORIES

- Marked ω -categories are ω -categories with a specified class of marked cells closed under composition and coherences.
- We can build a semantics of ICaTT in marked ω -categories:

$$\begin{array}{ccc} \text{Mod}(\text{ICaTT}) & \xrightarrow{L} & \text{Cat}_\omega^+ \\ & \searrow J^* & \downarrow U \\ & & \text{Cat}_\omega \end{array}$$

by marking a cell if it factors through the walking equivalence.

- We have constructed terms witnessing that LX is *fibrant*:
 - marked cells are invertible
 - their inverses are marked
 - marked cells are closed under (marked) equivalence
 - marked cells satisfy the 2-of-6 property

CONCLUSIONS

- We have extended CaTT with a new type $\text{Inv}(t)$ for invertible terms.

CONCLUSIONS

- We have extended CaTT with a new type $\text{Inv}(t)$ for invertible terms.
- We have shown that the extension is conservative.

CONCLUSIONS

- We have extended CaTT with a new type $\text{Inv}(t)$ for invertible terms.
- We have shown that the extension is conservative.
- We have given a simple syntactic presentation of the walking equivalence.

CONCLUSIONS

- We have extended CaTT with a new type $\text{Inv}(t)$ for invertible terms.
- We have shown that the extension is conservative.
- We have given a simple syntactic presentation of the walking equivalence.
- We have given semantics in (fibrant) marked ω -categories.

CONCLUSIONS

- We have extended CaTT with a new type $\text{Inv}(t)$ for invertible terms.
 - We have shown that the extension is conservative.
 - We have given a simple syntactic presentation of the walking equivalence.
 - We have given semantics in (fibrant) marked ω -categories.
-
- We have implemented a type checker for ICaTT in OCaml, and used it to verify the proof of fibrancy.






CONCLUSIONS

- We have extended CaTT with a new type $\text{Inv}(t)$ for invertible terms.
 - We have shown that the extension is conservative.
 - We have given a simple syntactic presentation of the walking equivalence.
 - We have given semantics in (fibrant) marked ω -categories.
-
- We have implemented a type checker for ICaTT in OCaml, and used it to verify the proof of fibrancy.

CONCLUSIONS

- We have extended CaTT with a new type $\text{Inv}(t)$ for invertible terms.
 - We have shown that the extension is conservative.
 - We have given a simple syntactic presentation of the walking equivalence.
 - We have given semantics in (fibrant) marked ω -categories.
-
- We have implemented a type checker for ICaTT in OCaml, and used it to verify the proof of fibrancy.

Thank you for your attention!

-  Benjamin, Thibaut and Ioannis Markakis (2024). *Invertible cells in ω -categories*. Pre-published.
-  Finster, Eric and Samuel Mimram (2017). “A type-theoretical definition of weak ω -categories”. In: *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2017)*. Reyjavik Iceland: ACM.
-  Fujii, Soichiro, Keisuke Hoshino, and Yuki Maehara (2025). *ω -Equifibrations between Strict and Weak ω -Categories*. Pre-published.
-  Hoshino, Keisuke, Soichiro Fujii, and Yuki Maehara (2024). “Weakly invertible cells in a weak ω -category”. In: *Higher Structures* 8.2, pp. 386–415.
-  Ozornova, Viktoriya and Martina Rovelli (2024). “What Is an Equivalence in a Higher Category?” In: *Bulletin of the London Mathematical Society* 56.1, pp. 1–58.